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A note on reward-risk portfolio selection and two-fund separation

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Abstract

This paper presents a general reward-risk portfolio selection model and derives sufficient conditions for two-fund separation. In particular we show that many reward-risk models presented in the literature satisfy these conditions.

Keywords: Two-fund separation, reward-risk preferences.

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1 Introduction

The modern portfolio theory of Markowitz (1952) is a rich source of intuition and also the basis for many practical decisions. Mean-variance agents may differ with respect to the degree they are willing to trade off mean against variance but all will choose from the set of efficient portfolios, those which maximize mean given a constraint on variance. Moreover, under certain conditions, the mean-variance model of portfolio selection leads to two-fund separation (Tobin 1958), i.e., all agents hold a combination of the *same* portfolio of risky assets combined with the risk-free asset. Two-fund separation greatly simplifies the advice one should give to a heterogenous set of agents since the proportion of risky assets in the optimal portfolio is independent from agent's risk aversion. Moreover, it implies a simple asset pricing structure in which a single risk factor explains the rewards agents get in equilibrium.

We derive sufficient conditions for two-fund separation in a general reward-risk model, where agents' preferences are assumed to be increasing functions of a reward measure and decreasing functions of a risk measure. We show that two-fund separation holds if reward and risk measures can be transformed by means of strictly increasing functions into positive homogeneous, translation invariant or translation equivariant functionals. In this case, the efficient frontier is a straight line in the transformed reward-risk diagram.

Several reward and risk measures introduced in the literature satisfy the conditions for two-fund separation. Mean and variance, semi-variance (Markowitz 1959), lower partial moments (Bawa and Lindenberg 1977, Fishburn 1977, Harlow and Rao 1989), the Gini measure (Yitzhaki 1982), general deviation measures (Rockafellar, Uryasev, and Zabarankin 2006), etc. Since many of these measures are defined based only on few principles of rationality, it follows that two-fund separation is a common property to many rational mean-risk models, including even those which are consistent with second order stochastic dominance (De Giorgi 2005). This result is surprising because strong conditions on agents' utility functions are needed in order to obtain two-fund separation within expected utility theory (Cass and Stiglitz 1970).

In Section 2 we introduce the general reward-risk model and derive our main result. Examples are discussed in Section 3. All proofs are given in the Appendix.

2 General reward-risk model and two-fund separation

We consider a two-period economy. The set S of states of the world in the second period is endowed with a sigma-algebra Σ . An element $s \in S$ is an individual state of the world, while $A \in \Sigma$ is an event. A random variable on (S, Σ) is a real-valued function $X : S \rightarrow \mathbb{R}$ such that $X^{-1}(J) \in \Sigma$ for all intervals $J \subset \mathbb{R}$. The space of random variables on (S, Σ) is denoted by $\mathcal{L}_0(S, \Sigma)$. We don't assume existence of a probability measure on (S, Σ) , since our conditions for two-fund separation do not require any assumption on the *distribution* of

random variables.

There are $K + 1$ assets with random payoffs $A_k \in \mathcal{L}_0(S, \Sigma)$ and prices q_k , $k = 0, \dots, K$. Asset 0 is the risk-free asset with $A_0 = 1$, $q_0 > 0$, and gross return $R_0 = 1/q_0$. We assume that assets can be traded without restrictions: the marketed subspace is denoted by $\mathcal{X} = \{\sum_{k=0}^K \theta_k A_k | (\theta_0, \dots, \theta_K)' \in \mathbb{R}^{K+1}\}$. An element $X \in \mathcal{X}$ is called a portfolio. For $X = \sum_{k=0}^K \theta_k A_k \in \mathcal{X}$ we denote by $q(X) = \sum_{k=0}^K \theta_k q_k$ the price of portfolio X .

We introduce two real-valued functions $\mu : \mathcal{X} \rightarrow \mathbb{R}$ and $\rho : \mathcal{X} \rightarrow \mathbb{R}$. We call μ a *reward* measure and ρ a *risk* measure. It is plausible to assume that $\mu(X) \geq \mu(Y)$ and $\rho(X) \leq \rho(Y)$ when $X(s) \geq Y(s)$ for all $s \in S$. We don't make these monotonicity assumptions here since they are not necessary for our main result and they would exclude from our framework the mean-variance model (variance violates monotonicity). The basic properties of reward and risk measures we consider are the following.

Definition 1. Let $\zeta : \mathcal{X} \rightarrow \mathbb{R}$ be a real-valued function on \mathcal{X} (ζ is a reward or a risk measure). We say that ζ is:

(i) (positive) homogeneous of degree γ if

$$\zeta(\kappa X) = |\kappa|^\gamma \zeta(X)$$

for all $\kappa \in \mathbb{R}$ ($\kappa \geq 0$) and $X \in \mathcal{X}$,

(ii) translation invariant if

$$\zeta(X + a) = \zeta(X)$$

for all $a \in \mathbb{R}$ and $X \in \mathcal{X}$,

(iii) translation equivariant if

$$\zeta(X + a) = \zeta(X) + a$$

for all $a \in \mathbb{R}$ and $X \in \mathcal{X}$, or

$$\zeta(X + a) = \zeta(X) - a$$

for all $a \in \mathbb{R}$ and $X \in \mathcal{G}$. In the first case, we say that ζ is positive translation equivariant, while in the second case it is negative translation equivariant.

When reward or risk measures are translation invariant, then adding a risk-free position to the portfolio doesn't change risk or reward. By contrast, translation equivariant implies that adding a risk-free position increases (positive translation equivariant) or decreases (negative translation equivariant) reward or risk by the same amount as the risk-free addition. Negative translation equivariance is often assumed for risk measures that are interpreted as risk capital requirement; see Föllmer and Schied (2002). Positive translation equivariance and translation

invariance are satisfied by many reward and risk measures used in reward-risk models for portfolio selection.

An agent has initial wealth w_0 and wants to choose from \mathcal{X} . She possesses a utility function $U : \mathcal{X} \rightarrow \mathbb{R}$ which satisfies the following assumption:

Assumption 1 $((\mu, \rho)$ -preferences).

$$U(X) = v(\mu(X), \rho(X))$$

for all $X \in \mathcal{X}$ where $v : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(\mu, \rho) \rightarrow v(\mu, \rho)$ is continuously differentiable, strictly increasing in μ , strictly decreasing in ρ , and concave.

Assumption 1 simply states that agents evaluate random payoffs only through the reward measure μ and the risk measure ρ , respectively, and are risk averse. Preferences according to Assumption 1 are called *general reward-risk preferences* (or simply (μ, ρ) -preferences).

The agent solves the following decision problem

$$\max_{X \in \mathcal{X}} U(X) \text{ such that } q(X) \leq w_0, \quad (1)$$

i.e., she maximizes her utility given the budget constraint.

Given the purpose of this paper to characterize solutions of the maximization of reward-risk preferences we make the following additional assumption:

Assumption 2 (Existence of solutions with bounded reward and risk). *Problem (1) possesses a solution $X^* \in \mathcal{X}$ with $\mu(X^*) < \infty$ and $\rho(X^*) < \infty$.*

The following holds:

Proposition 1. *Under Assumptions 1 and 2, X^* solves Problem (1) if and only if there exists $\bar{\rho} \in \mathbb{R}$ such that X^* solves*

$$\max_{X \in \mathcal{X}} \mu(X) \text{ such that } \rho(X) \leq \bar{\rho} \text{ and } q(X) \leq w_0. \quad (2)$$

We are now ready to prove our main result:

Theorem 1 (Two-fund separation). *Let $\mu : \mathcal{X} \rightarrow \mathbb{R}$ and $\rho : \mathcal{X} \rightarrow \mathbb{R}$ be a reward and risk measure, respectively, on \mathcal{X} . Suppose that strictly increasing transformations $T_\mu : \mathbb{R} \rightarrow \mathbb{R}$, $T_\rho : \mathbb{R} \rightarrow \mathbb{R}_+$ exist such that $\tilde{\mu} = T_\mu \circ \mu$ and $\tilde{\rho} = T_\rho \circ \rho$ are positive homogeneous of degree 1, and translation invariant or translation equivariant. Then (μ, ρ) -preferences satisfy two-fund separation, i.e., there exists $X^* \in \mathcal{X}$, such that for all utility functions U that satisfy Assumptions 1 and 2 with reward measure μ and risk measure ρ , there exist parameters $\alpha^U, \alpha_0^U \in \mathbb{R}$, $\alpha^U \geq 0$, such that $\alpha^U X^* + \alpha_0^U$ solves Problem (1).*

Theorem 1 states that two-fund separation holds in general reward-risk models when the efficient frontier is a straight line in the reward-risk diagram, *after* transforming reward and risk measures by means of strictly increasing transformations. Increasing transformations change the risk-reward trade-off but not the ranking of asset payoffs, thus the transformed reward-risk model delivers the same efficient frontier as the original reward-risk model.

Many reward and the risk measures are themselves positive homogeneous so that it makes sense to emphasize this case.

Corollary 1. *Let $\mu : \mathcal{X} \rightarrow \mathbb{R}$ and $\rho : \mathcal{X} \rightarrow \mathbb{R}_+$ be reward and risk measures, respectively, on \mathcal{X} . Suppose that one of the following two properties holds for μ and ρ :*

- (i) *nonnegative on \mathcal{X} , translation invariant and (positive) homogeneous of degree $\gamma > 0$,*
- (ii) *translation equivariant and (positive) homogeneous of degree 1.*

Then under Assumption 2, (μ, ρ) -preferences satisfy two-fund separation.

3 Examples

In this section we provide examples of reward-risk models that satisfies two-fund separation. We will assume that (S, Σ) is endowed with a probability measure \mathbb{P} and a $X \in \mathcal{L}_0(S, \Sigma)$ has cumulative distribution function $F_X(x) = \mathbb{P}[X \leq x]$ for all x .

Example 1 (Mean-variance model (Markowitz 1952)). *We observe that the mean is positive homogenous and positive translation equivariant, and that the variance is nonnegative, translation invariant, and positive homogeneous of degree 2.*

Example 2 (Mean-Semi-variance model (Markowitz 1959)). *For a random variable $X \in \mathcal{L}_0(S, \Sigma)$ the semi-variance is defined as*

$$sv(X) = \int_{\mathbb{R}} [\min(0, x - \mathbb{E}[X])]^2 dF_X(x).$$

Semi-variance is nonnegative, translation invariant, and positive homogenous of degree 2.

Example 3 (Mean-lower partial moment model (Bawa and Lindenberg 1977)). *For a random variable $X \in \mathcal{L}_0(S, \Sigma)$ the lower partial moment with target wealth $\tau : \mathcal{X} \rightarrow \mathbb{R}$ is defined by:*

$$LPM(X; \alpha, \tau) = \int_{\mathbb{R}} [\max(0, \tau(X) - x)]^\alpha dF_x(x) \quad (3)$$

where $\alpha \geq 1$. Here we assume that the target $\tau(X)$ might depend on X . One example of target function is the risk-free return, i.e., $\tau(X) = R_0 q(X)$ corresponds to the final wealth when the price $q(X)$ of portfolio X is invested on the risk-free asset instead of X . Another

example is the expected return, i.e., $\tau(X) = \mathbb{E}[X]$. In these two cases τ is positive translation equivariant and positive homogeneous of degree 1. Consequently, $LPM(\cdot; \alpha, \tau)$ is translation invariant and positive homogeneous of degree α .

Example 4 (Mean-Gini model (Yitzhaki 1982)). For a random variable $X \in \mathcal{L}_0(S, \Sigma)$ the Gini measure is defined as

$$\Gamma(X) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y| dF_X(x) dF_Y(y).$$

The Gini measure is nonnegative, positive homogenous, and translation invariant.

Example 5 (Mean-General deviation measures model (Rockafellar et al. 2006)). Artzner, Delbaen, Eber, and Heath (1999) axiomatize the class of coherent risk measures. Coherent risk measures are per definition positive homogenous, negative translation equivariant, but can also take negative values. Rockafellar et al. (2006) introduce the class of general deviation measure that correspond to coherent risk measures when these are applied to deviations $X - \mathbb{E}[X]$ instead of random variables X . General deviation measures are nonnegative, positive homogeneous and translation invariant.

Example 6 (Behavioral reward-risk model (De Giorgi, Hens, and Mayer 2006)). The reward measure CPT^+ is the prospect utility on gains, while the risk measure CPT^- is the prospect utility on losses. For a random variable $X \in \mathcal{L}_0(S, \Sigma)$ we have:

$$CPT^+(X; \tau) = - \int_{\mathbb{R}} v(\max(0, x - \tau(X))) d(1 - w(F_X(x)))$$

and

$$CPT^-(X; \tau) = -\frac{1}{\lambda} \int_{\mathbb{R}} v(-\min(0, x - \tau(X))) d(w(F_X(x)))$$

where v is twice-differentiable on $\mathbb{R} \setminus \{0\}$, strictly increasing, $v(0) = 0$, convex on $(-\infty, 0)$ and concave on $(0, \infty)$ and $\lambda = \lim_{x \nearrow 0} v(x) / \lim_{x \searrow 0} v'(x) \geq 1$ is the index of loss aversion. The function $\tau : \mathcal{X} \rightarrow \mathbb{R}$ is a target function and might depend on X . The cumulative prospect theory value function of Tversky and Kahneman (1992) corresponds to

$$CPT(X; \tau) = CPT^+(X; \tau) - \lambda CPT^-(X; \tau).$$

It can be easily shown that $CPT^+(\cdot; \tau)$ and $CPT^-(\cdot; \tau)$ are nonnegative, positive homogeneous, and translation invariant, when v is positive homogenous on $x < 0$ and $x > 0$, and the target function is positive homogenous and positive translation equivariant.

The piecewise-power function suggested by Tversky and Kahneman (1992)

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0 \end{cases}$$

for $\lambda \geq 1$ and $\alpha, \beta \in (0, 1)$, is positive homogenous both on $x < 0$ and $x > 0$. Moreover, the reference point $\tau(X) = R_0 q(X)$ (see Example 3) is positive homogeneous and positive translation equivariant. Thus, the behavioral reward-risk model where v is the piecewise-power function of Tversky and Kahneman (1992) and the reference point is the risk-free return, satisfies two-fund separation. This observation generalizes to any distribution the result of Barberis and Huang (2008), who show that two-fund separation holds with cumulative prospect theory preferences, when all assets are assumed to be normally distributed.

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A Proofs

A.1 Proof of Proposition 1

Let X^* be a solution to Problem (1) with $\rho(X^*) < \infty$ and set $\bar{\rho} = \rho(X^*)$. Suppose that $X \in \mathcal{X}$ exists with $q(X) \leq w_0$, $\rho(X) \leq \bar{\rho}$ and $\mu(X) > \mu(X^*)$. Since U is strictly increasing in μ and strictly decreasing in ρ , then $U(X) > U(X^*)$, a contradiction to the optimality of X^* for Problem (1).

On the other hand, let $X^*(\bar{\rho})$ be the optimal solution to Problem (2), when the upper bound for ρ is $\bar{\rho} \in \mathbb{R}$. Assume that we find X^* such that $q(X^*) \leq w_0$ and $U(X^*) > U(X^*(\bar{\rho}))$ for all $\bar{\rho} \in \mathbb{R}$. Then $\rho(X^*) = \infty$, a contradiction to Assumption 2. If we find $\bar{\rho} \in \mathbb{R}$, such that $\rho(X^*) = \bar{\rho}$, then $\mu(X^*(\bar{\rho})) \geq \mu(X^*)$. Moreover, without loss of generality, we can assume that $\rho(X^*(\bar{\rho})) = \bar{\rho}$. Therefore, since U strictly increases in μ and strictly decreases in ρ , we must have $U(X^*(\bar{\rho})) = U(X^*)$, a contradiction.

A.2 Proof of Theorem 1

Proposition 1 implies that Problems (1) and (2) are equivalent. We thus consider solutions to Problem (2). Since T_μ and T_ρ are strictly increasing, Problem (2) is equivalent to

$$\max_{X \in \mathcal{X}} \tilde{\mu}(X) \text{ such that } \tilde{\rho}(X) \leq \hat{\rho} \text{ and } q(X) \leq w_0 \quad (4)$$

where $\hat{\rho} = T_\rho(\bar{\rho}) \geq 0$.

Let \hat{X} be a solution to Problem (4) with risk constraint $\hat{\rho} > 0$, and initial wealth \hat{w}_0 . We assume without loss of generality that $\tilde{\rho}(\hat{X}) = \hat{\rho}$. Consider an agent with risk constraint

$\check{\rho} > 0$, and initial wealth $\check{w}_0 > 0$. She solves

$$\max_{X \in \mathcal{X}} \tilde{\mu}(X) \text{ such that } \tilde{\rho}(X) \leq \check{\rho} \text{ and } q(X) \leq \check{w}_0.$$

Assume that $\tilde{\mu}$ and $\tilde{\rho}$ are positive homogenous of degree 1 and translation invariant. Then for $\alpha \geq 0$ and $\alpha_0 \in \mathbb{R}$ we have

$$\tilde{\rho}(\alpha \hat{X} + \alpha_0) = \alpha \tilde{\rho}(\hat{X}) \text{ and } q(\alpha \hat{X} + \alpha_0) = \alpha q(\hat{X}) + \alpha_0 q_0.$$

Set $\check{\alpha} = \check{\rho}/\hat{\rho} > 0$ and $\check{\alpha}_0 = (\check{w}_0 - (\check{\rho}/\hat{\rho}) q(\hat{X}))/q_0$. Then

$$\tilde{\rho}(\check{\alpha} \hat{X} + \check{\alpha}_0) = \check{\rho} \text{ and } q(\check{\alpha} \hat{X} + \check{\alpha}_0) = \check{w}_0.$$

Assume that $\check{Y} \in \mathcal{X}$ exists with $\tilde{\rho}(\check{Y}) \leq \check{\rho}$, $q(\check{Y}) \leq \check{w}_0$ and $\mu(\check{Y}) > \mu(\check{\alpha} \hat{X} + \check{\alpha}_0)$. Let $\hat{Y} = (1/\check{\alpha})(\check{Y} - \check{\alpha}_0)$. Since $\tilde{\mu}$ and $\tilde{\rho}$ are positive homogeneous of degree 1 and translation invariant, we have

$$\begin{aligned} \tilde{\mu}(\hat{Y}) &= (1/\check{\alpha}) \mu(\check{Y}) > \tilde{\mu}(\hat{X}), \\ \tilde{\rho}(\hat{Y}) &= \frac{1}{\check{\alpha}} \tilde{\rho}(\check{Y}) \leq \frac{1}{\check{\rho}/\hat{\rho}} \check{\rho} = \hat{\rho}, \end{aligned}$$

and

$$q(\hat{Y}) = \frac{1}{\check{\alpha}} q(\check{Y}) - \frac{\check{\alpha}_0}{\check{\alpha}} q_0 \leq \frac{1}{\check{\alpha}} \check{w}_0 - \frac{\check{\alpha}_0}{\check{\alpha}} q_0 = q(\hat{X}) \leq \hat{w}_0.$$

This contradicts the optimality of \hat{X} when the risk constraint is $\hat{\rho}$ and the budget restriction is \hat{w}_0 . Thus, $\check{X} = \check{\alpha} \hat{X} + \check{\alpha}_0$ is optimal when the risk constraint is $\check{\rho}$ and the budget restriction is \check{w}_0 .

When $\check{\rho} = 0$, assume that $\check{X} \in \mathcal{X}$ exists, $\check{X} \neq \alpha_0$ for all $\alpha_0 \in \mathbb{R}$, with $\tilde{\rho}(\check{X}) = 0$ and $\tilde{\mu}(\check{X}) > 0$. Then for all $\alpha, \alpha_0 \in \mathbb{R}$, $\alpha > 0$, we have $\tilde{\rho}(\alpha \check{X} + \alpha_0) = \alpha \tilde{\rho}(\check{X}) = 0$, $\tilde{\mu}(\alpha \check{X} + \alpha_0) = \alpha \tilde{\mu}(\check{X})$, and $q(\alpha \check{X} + \alpha_0) = \alpha q(\check{X}) + \alpha_0 q_0$. Let $\alpha_0 = (\check{w}_0 - \alpha q(\check{X}))/q_0$ and let $\alpha \rightarrow \infty$, then $\tilde{\rho}(\alpha \check{X} + \alpha_0) = 0$, $\tilde{\mu}(\alpha \check{X} + \alpha_0) \rightarrow \infty$, and $q(\alpha \check{X} + \alpha_0) = \check{w}_0$. A contradiction to Assumption 2. When $\check{\rho} = 0$, then any $\check{X} = \alpha_0 = 0 \hat{X} + \alpha_0$ is optimal.

The proof for the case where μ or ρ are translation equivariant is almost identical.

A.3 Proof of Corollary 1

In (i), let $\gamma_\mu > 0$, $\gamma_\rho > 0$ be the degree of homogeneity of μ and ρ , respectively. Take the strictly increasing transformations $T_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+, x \rightarrow x^{\frac{1}{\gamma_\mu}}$ and $T_\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+, x \rightarrow x^{\frac{1}{\gamma_\rho}}$.

In (ii), the reward or risk measure is already positive homogeneous of degree 1 and translation equivariant. We can take $T_\mu : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow x$ and $T_\rho : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow x$.

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